INDIAN STATISTICAL INSTITUTE
Semestral Exam
Algebra-I
2018-2019

Total marks: 100
Time: 3 hours

Question **1** is compulsory. Answer any **4** questions from the rest.

1. State true or false. Justify your answers.
   (a) $\mathbb{Z}_n$ has even number of generators for $n > 2$.
   (b) S_4 is isomorphic to D_{24}.
   (c) If G is non-abelian then $\mathrm{Aut}(G)$ is not cyclic.
   (d) Q_8, the group of quaternions, can be written as a semidirect product of its proper subgroups. (4 × 5)

2. (i) How many cyclic subgroups does $\mathbb{Z}/15\mathbb{Z}$ have? List them.
   (ii) (a) Find the orders of the elements $a = 2^{n-1} + 1$ and $b = 2^{n-1} - 1$ in $(\mathbb{Z}/2^n\mathbb{Z})^\times$ for $n \geq 3$.
   (b) Hence show that $(\mathbb{Z}/2^n\mathbb{Z})^\times$ is not a cyclic group for $n \geq 3$. (8+8+4)

3. (a) Let G be a group and H be a subgroup of G. Consider the left regular action of G on the set of all left cosets of H in G. Find the kernel of this action.
   (b) Let G be a group and H be a subgroup of finite index in G. Then show that there exists a normal subgroup N of G such that N is of finite index in G and N is contained in H. (8+12)

4. (a) Show that a group of order 56 has a normal Sylow p-subgroup for some prime p diving its order.
   (b) Find all normal subgroups of S_n for all $n \geq 5$. (You may use A_n is simple for all $n \geq 5$.) (8+12)

5. (a)) Define commutator subgroup $[G, G]$ of a group G.
   (b) Let H be a subgroup of G. Then show that H is normal in G and G/H is abelian if and only if $[G, G] \subseteq H$.
   (c) Find the commutator subgroups of S_n, for all $n \geq 3$. (2+8+10)

6. (a) Classify all groups of order p^2, where p is a prime.
   (c) Classify all groups of order pq, where p, q are primes, $p < q$. (8+12)

**** End ****